Biharmonic Steklov problems and spectral inequalities on differential forms

Rodolphe Abou Assali[1 2]

[1] Université de Lorraine, CNRS, IECL

[2] Université Libanaise, Faculté des Sciences II, Département de Mathématiques

Institut Élie Cartan de Lorraine

UNIVERSITÉ DE LORRAINE

Université Libanaise

Ecole Doctorale en Sciences et Technologie

Introduction

Kuttler and Sigillito [KS68] established spectral inequalities on bounded domains in $\mathbb{R}^2$ linking the eigenvalues of the standard Steklov, Dirichlet, and Neumann problems with those of the biharmonic Steklov problem with Dirichlet and Neumann boundary conditions. Hassannezhad and Siffert [HS20] later generalized some of these inequalities to compact star-shaped Riemannian manifolds in the scalar case. In this work, we extend certain Kuttler–Sigillito (K–S) inequalities to differential forms on compact Riemannian manifolds. To this end, we introduce a biharmonic Steklov problem with Neumann-type boundary conditions, prove its ellipticity, establish the existence of a discrete spectrum, and provide variational characterizations of its eigenvalues.

Classical eigenvalue problems for functions

Let (M^n, g) be a compact connected Riemannian manifold with boundary ∂M. We denote by:

- $\Delta = -\operatorname{div}\nabla$ the Beltrami Laplacian, $\Delta^2 = \Delta \circ \Delta$ the bi-Laplacian.
- ν the inward unit normal vector on ∂M.
- $\partial_\nu u$ the inner normal derivative.

Definition. *We consider the following spectral problems.*

Dirichlet problem

$$\begin{cases} \Delta u = \lambda u & \text{on } M \\ u = 0 & \text{on } \partial M. \end{cases}$$

Neumann problem

$$\begin{cases} \Delta u = \mu u & \text{on } M \\ \partial_\nu u = 0 & \text{on } \partial M. \end{cases}$$

Steklov problem

$$\begin{cases} \Delta u = 0 & \text{on } M \\ \partial_\nu u = -\sigma u & \text{on } \partial M. \end{cases}$$

Biharmonic Steklov Dirichlet (BSD)

$$\begin{cases} \Delta^2 u = 0 & \text{on } M \\ u = 0 & \text{on } \partial M \\ \Delta u - q\,\partial_\nu u = 0 & \text{on } \partial M. \end{cases}$$

Biharmonic Steklov Neumann (BSN)

$$\begin{cases} \Delta^2 u = 0 & \text{on } M \\ \partial_\nu u = 0 & \text{on } \partial M \\ \partial_\nu(\Delta u) + \ell u = 0 & \text{on } \partial M. \end{cases}$$

The parameters $\lambda, \mu, \sigma, q, \ell \in \mathbb{R}$ denote the corresponding eigenvalues.

Problem	Spectrum	Kernel
Dirichlet	$0 < \lambda_1 \leq \lambda_2 \leq \dots$	Trivial
Neumann	$0 \leq \mu_1 \leq \mu_2 \leq \dots$	Constant functions
Steklov	$0 \leq \sigma_1 \leq \sigma_2 \leq \dots$	Constant functions
BSD	$0 < q_1 \leq q_2 \leq \dots$	Trivial
BSN	$0 \leq \ell_1 \leq \ell_2 \leq \dots$	Constant functions

Kuttler–Sigillito inequalities in $\mathbb{R}^2$

Some Kuttler–Sigillito inequalities on a bounded planar domain with piecewise boundary [KS68].

- $\mu_1\sigma_1 \leq \ell_1$
- $q_1\sigma_1^2 \leq \ell_1$
- $\mu_1^{-1} \leq \lambda_1^{-1} + (q_1\ell_1)^{-\frac{1}{2}}$
- $\mu_1^{-1} \leq \lambda_1^{-1} + (q_1\sigma_1)^{-1}$

BSD for p–differential forms

We denote by $\Delta = d\delta + \delta d$ the Hodge-de Rham Laplacian on differential forms.

By [El +24], the following problem

$$(BSD)\begin{cases} \Delta^2\,\omega = 0 & \text{on } M \\ \omega = 0 & \text{on } \partial M \\ \nu \lrcorner \Delta\,\omega + q\iota^*\delta\,\omega = 0 & \text{on } \partial M \\ \iota^*\Delta\,\omega - q\nu\lrcorner d\,\omega = 0 & \text{on } \partial M, \end{cases}$$

has a discrete spectrum consisting of an unbounded monotonously non–decreasing sequence of positive eigenvalues of finite multiplicities $(q_{j,p})_{j\geq 1}$.

BSN for p–forms

Theorem 1. *[Abo25] The following boundary problems*

$$(BSN)\begin{cases} \Delta^2\omega = 0 & \text{on } M \\ \nu\lrcorner\omega = 0 & \text{on } \partial M \\ \nu\lrcorner d\omega = 0 & \text{on } \partial M \\ \nu\lrcorner\Delta\omega = 0 & \text{on } \partial M \\ \nu\lrcorner\Delta d\omega + \ell\iota^*\omega = 0 & \text{on } \partial M \end{cases}$$

$$(BSN2)\begin{cases} \Delta^2\omega = 0 & \text{on } M \\ \nu\lrcorner\omega = 0 & \text{on } \partial M \\ \nu\lrcorner d\omega = 0 & \text{on } \partial M \\ \iota^*\delta\omega = 0 & \text{on } \partial M \\ \nu\lrcorner\Delta d\omega + \ell^{BSN2}\iota^*\omega = 0 & \text{on } \partial M \end{cases}
\qquad
(BSN3)\begin{cases} \Delta^2\omega = 0 & \text{on } M \\ \nu\lrcorner\omega = 0 & \text{on } \partial M \\ \nu\lrcorner d\omega = 0 & \text{on } \partial M \\ \nu\lrcorner\Delta\omega + \ell^{BSN3}\iota^*\delta\omega = 0 & \text{on } \partial M \\ \nu\lrcorner\Delta d\omega + \ell^{BSN3}\iota^*\omega = 0 & \text{on } \partial M. \end{cases}$$

has a discrete spectrum consisting of an unbounded non-decreasing sequence of positive real eigenvalues with finite multiplicities $(\ell_{i,p})_{i\geq 1}$. The kernel of this problem is $H_A^p(M)$, where the absolute de Rham cohomology of order p is the space

$$H_A^p(M) = \{\omega \in \Omega^p(M) \mid d\omega = 0 \text{ on } M,\ \delta\omega = 0 \text{ on } M,\ \nu\lrcorner\omega = 0 \text{ on } \partial M\}.$$

The first non-zero eigenvalue is given by:

$$\ell_{1,p} = \inf\left\{ \frac{\|\nu\lrcorner d\omega\|^2_{L^2(\partial M)}}{\|\omega\|^2_{L^2(M)}} \;\middle|\; \omega \in \Omega^p(M),\ \Delta\omega = 0 \text{ on } M,\ \nu\lrcorner\omega = 0 \text{ on } \partial M,\ \omega \perp_{L^2(M)} H_A^p(M) \right\}$$

$$= \inf\left\{ \frac{\|\Delta\omega\|^2_{L^2(M)}}{\|\iota^*\omega\|^2_{L^2(\partial M)}} \;\middle|\; \nu\lrcorner d\omega = 0,\ \nu\lrcorner\omega = 0 \text{ and } \iota^*\omega \neq 0 \text{ on } \partial M,\ \omega \perp_{L^2(\partial M)} H_A^p(M) \right\}.$$

K-S inequalities for p–forms

Theorem 2. *[Abo25] Let (M^n, g) be a compact Riemannian manifold with a smooth boundary ∂M.*

For $k \geq 1$, we have the following inequalities:

- $\mu_{k,p}\sigma_{1,p} \leq \ell_{k,p}$
- $\mu_{1,p}\sigma_{k,p} \leq \ell_{k,p}$

Moreover, the inequality is strict for $k = 1$.

- $q_{1,p}\sigma_{1,p}^2 < \ell_{1,p}$
- $\mu_{1,p}^{-1} < \lambda_{1,p}^{-1} + (q_{1,p}\ell_{1,p})^{-1/2}$
- $\mu_{1,p}^{-1} < \lambda_{1,p}^{-1} + (q_{1,p}\sigma_{1,p})^{-1}$

Here $\lambda_{k,p}, \mu_{k,p}, \sigma_{k,p}, q_{k,p}$ and $\ell_{k,p}$ denote the k^{th} eigenvalues of the Dirichlet, Neumann, Steklov, BSD, and BSN problems, respectively.

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